Lecture 7

Visualization, Randomness, and Computation

History of Data Science, Spring 2022 @ UC San Diego

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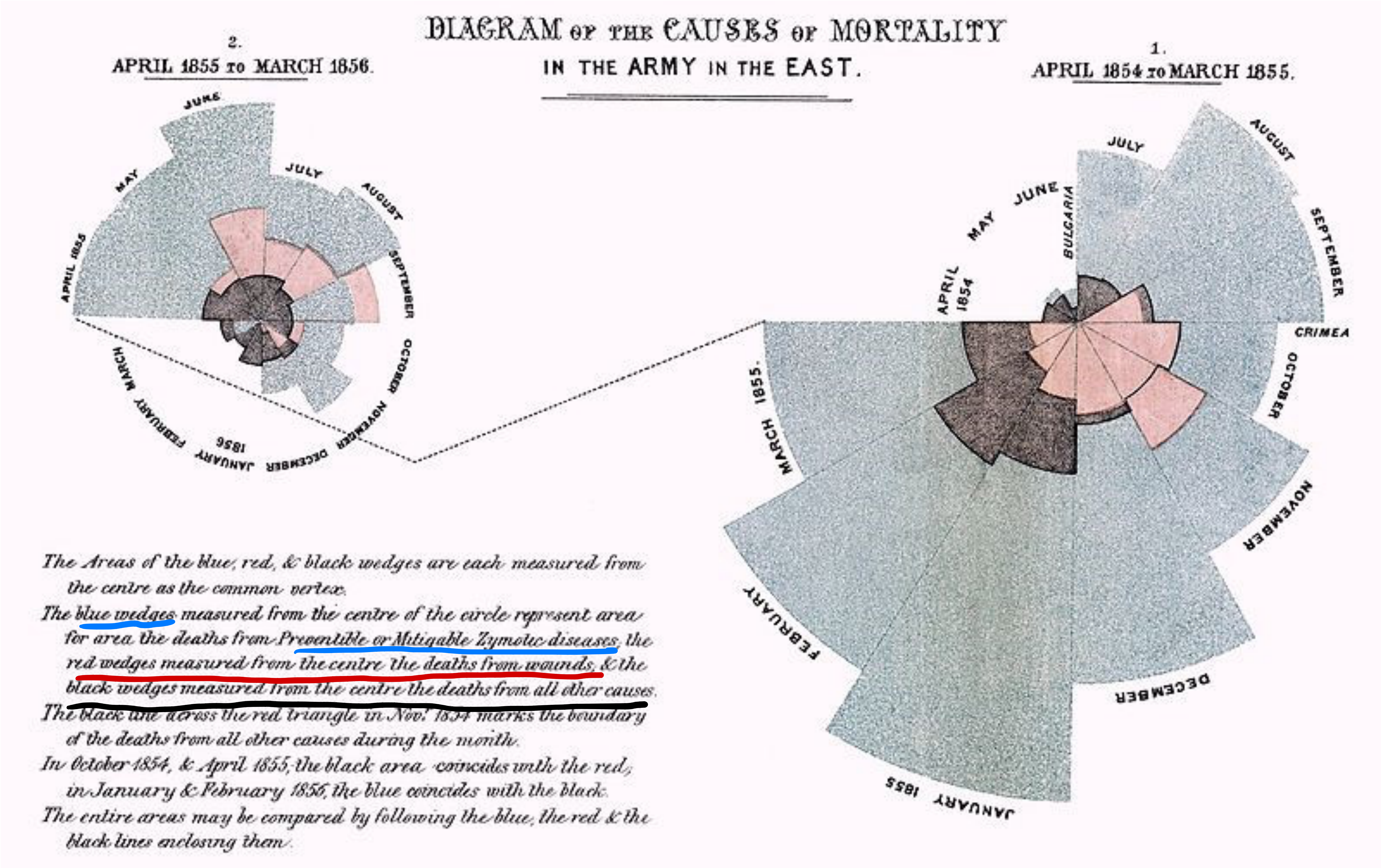
Announcements

- Look at others' Homework 6 websites!
  - Posted in the #homework6 channel on Slack.
  - If your images don't load, please fix them ASAP, and message if you need help.
- Homework 7 will be released by **noon tomorrow**.
- **Make sure to read Homework 5 solutions (posted on ~~Campuswire~~ Slack)!**
  - Many misconceptions about the Law of Large Numbers.

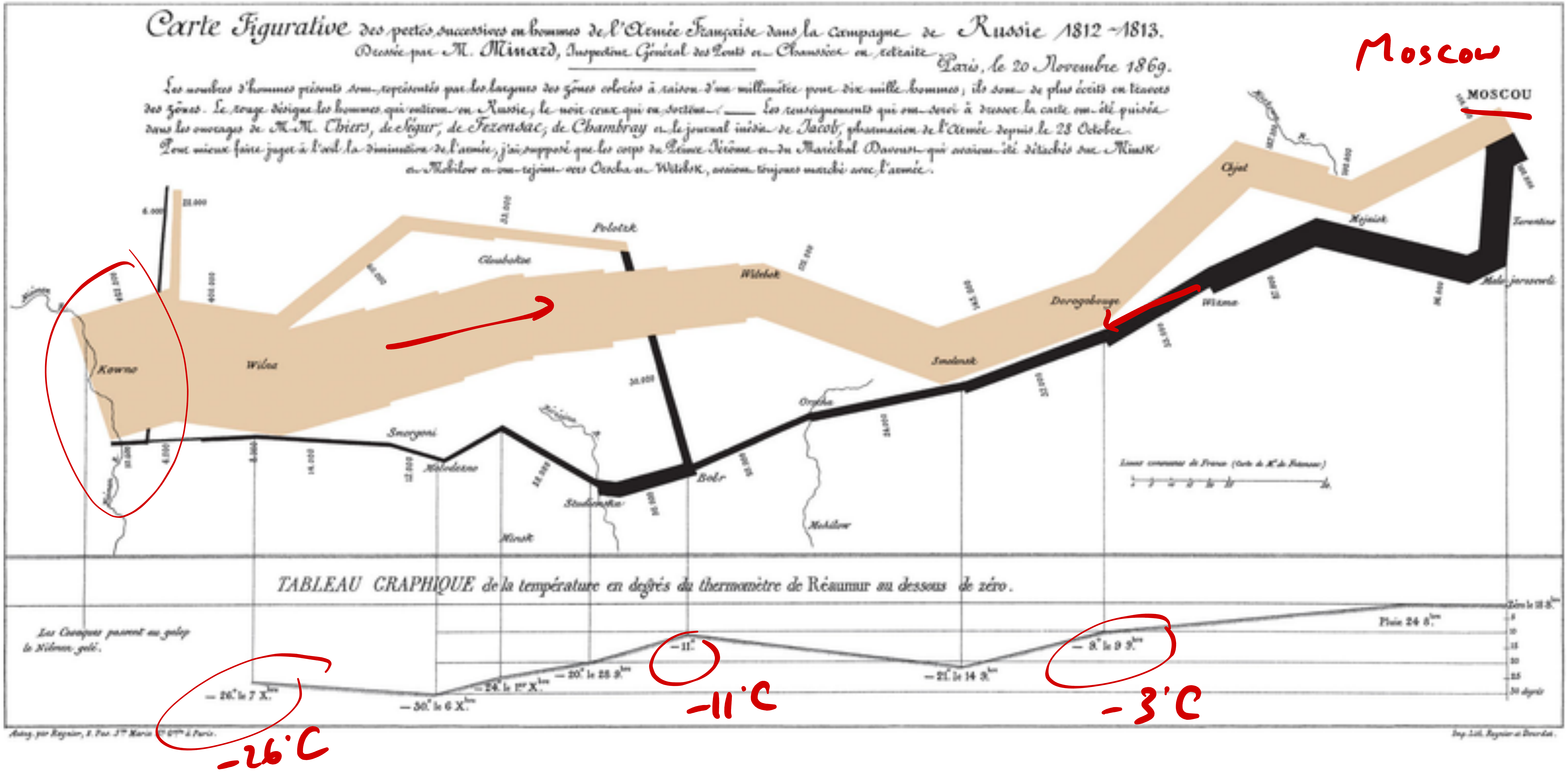
Agenda

- Finish visualization.
- Revisit the development of the Gaussian distribution.
- Abacus and the binary number system.

Visualization

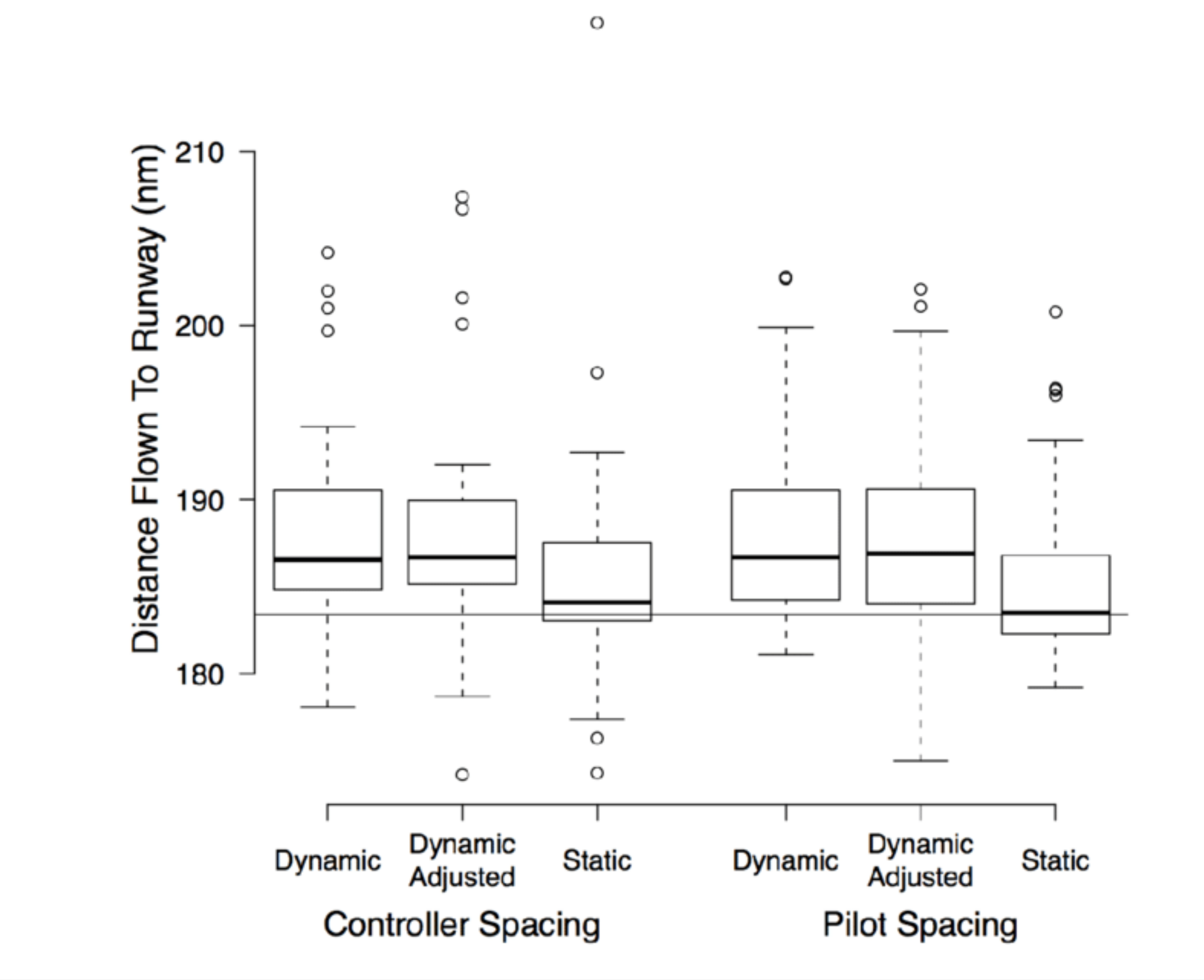


1855: Florence Nightingale's depiction of the deaths of British soldiers in the Crimean war. Florence Nightingale is known as the founder of modern nursing.



1869: Charles Joseph Minard's visualization of the French invasion of Russia (led by Napoleon).

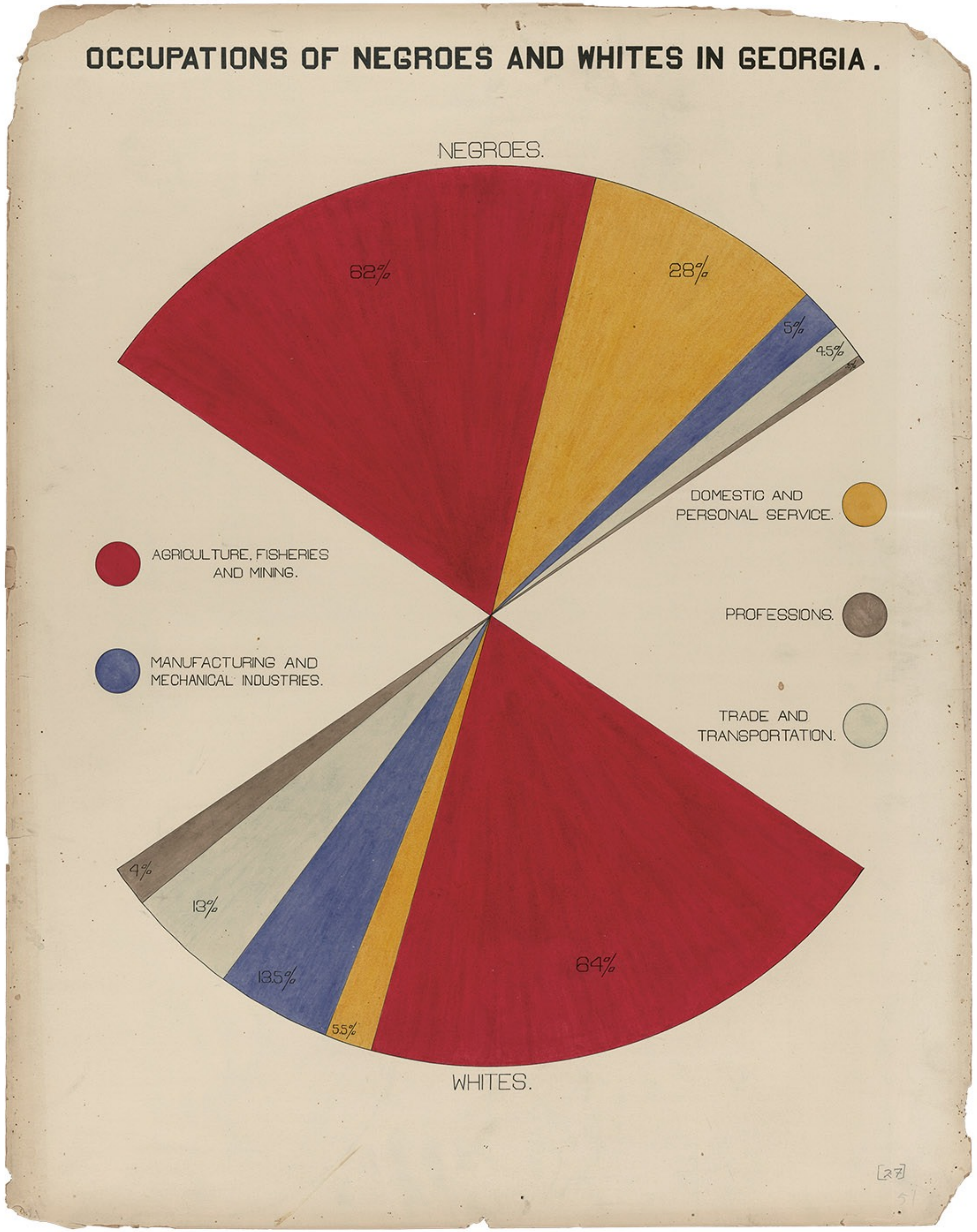
histogram alternative



1973: John Tukey, who defined the term "Exploratory Data Analysis", created the box plot, which describes a numerical distribution using a 5 number summary.

W. E. B. Dubois' visualizations of Black America

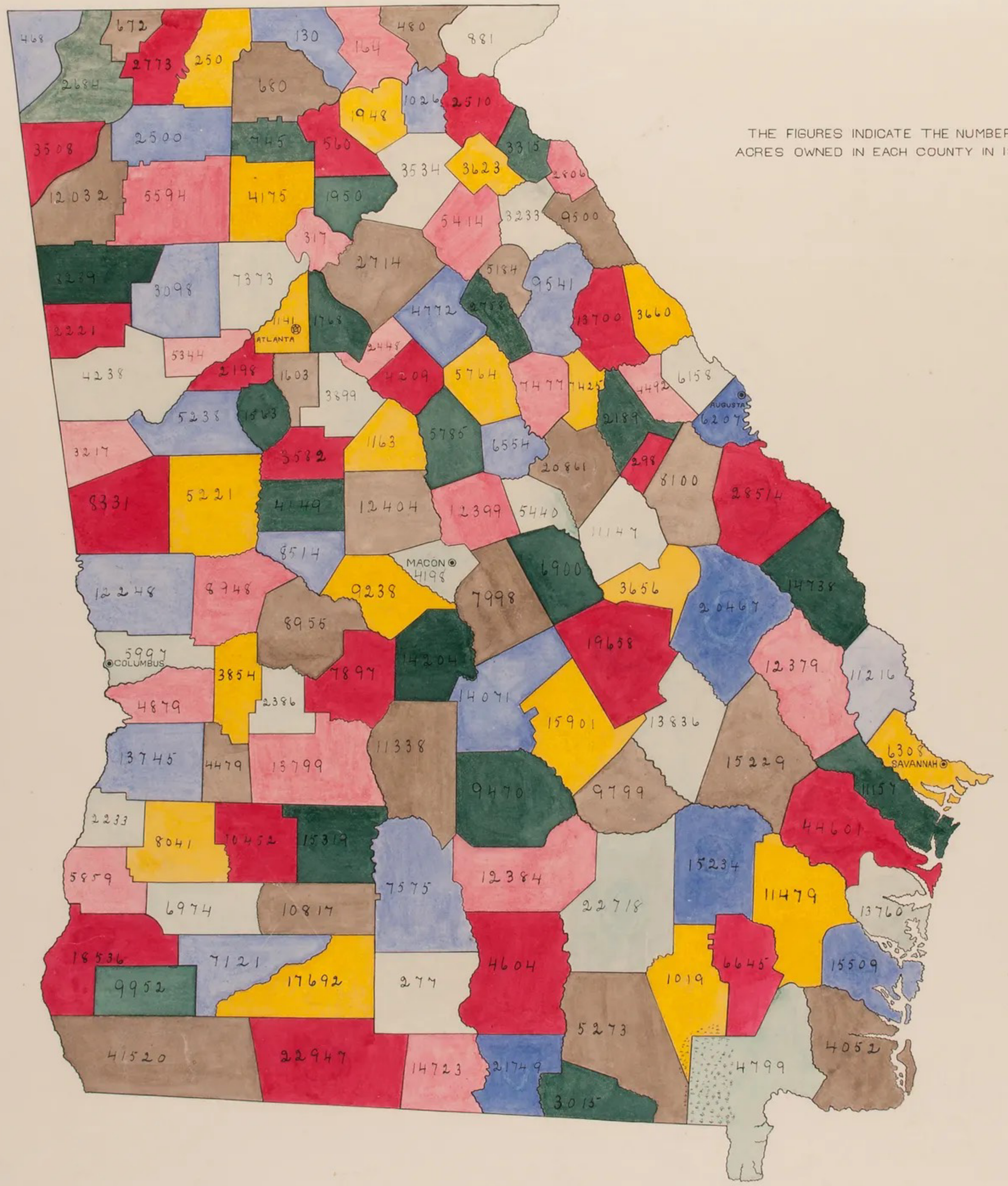
- W. E. B. Dubois (1868-1963) was an African-American historian, sociologist, and activist.
  - He helped create the National Association for the Advancement of Colored People.
  - He is also the first African-American to earn a PhD from Harvard.
- For the 1900 "Exposition Universelle" (world's fair) in Paris, he created a display celebrating the advancements of African-Americans.
  - Part of the display was a series of data visualizations (more in the reading).



https://lithub.com/w-e-b-du-bois-in-paris-the-exhibition-that-shattered-myths-about-black-america/

LAND OWNED BY NEGROES IN GEORGIA, U.S.A. 1870-1900.

THE FIGURES INDICATE THE NUMBER OF ACRES OWNED IN EACH COUNTY IN 1899.



Derivation of the Gaussian distribution

Gauss and least squares

- **Recall from Lecture 4:** one of the key differences between the approaches to least squares by Gauss and Legendre was that Gauss linked the theory of least squares to probability theory.

- Specifically, he posed the least squares **model** where

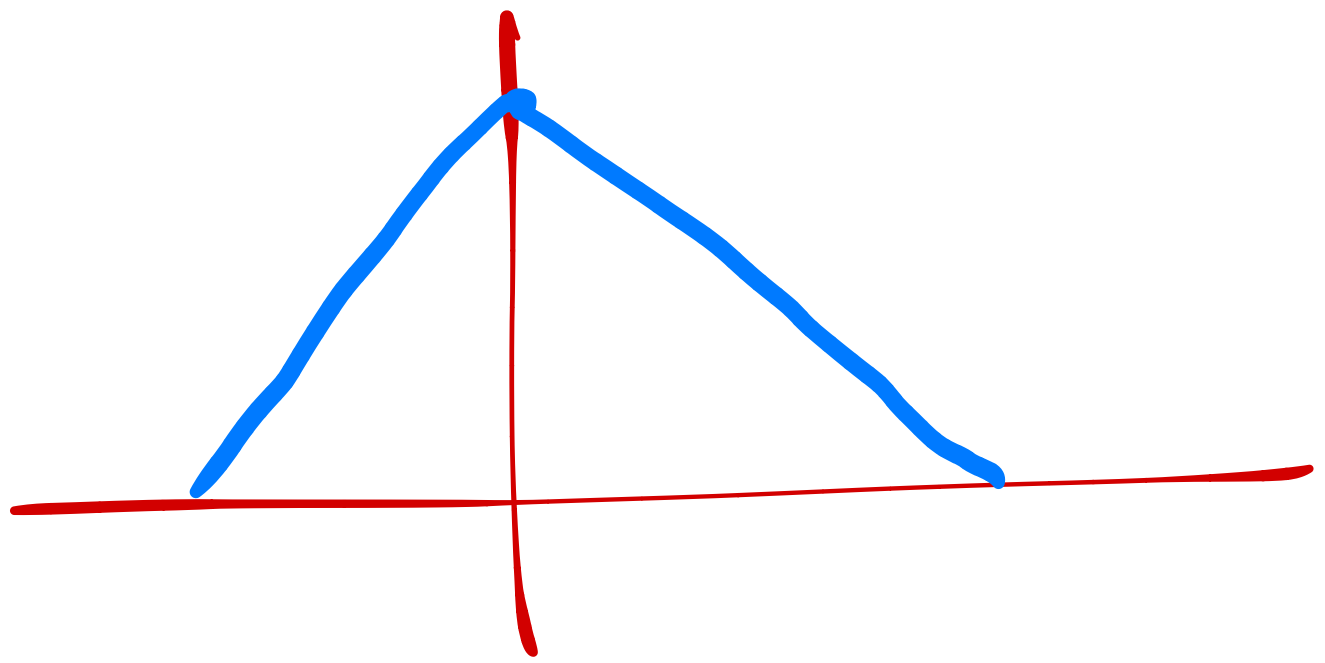
$$y_i = a + bx_i + \epsilon_i$$

where ϵ_i is a **random variable** that follows the following **error distribution**:

$$\phi(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Where did this distribution come from? 🤔

Criteria for the Gaussian distribution



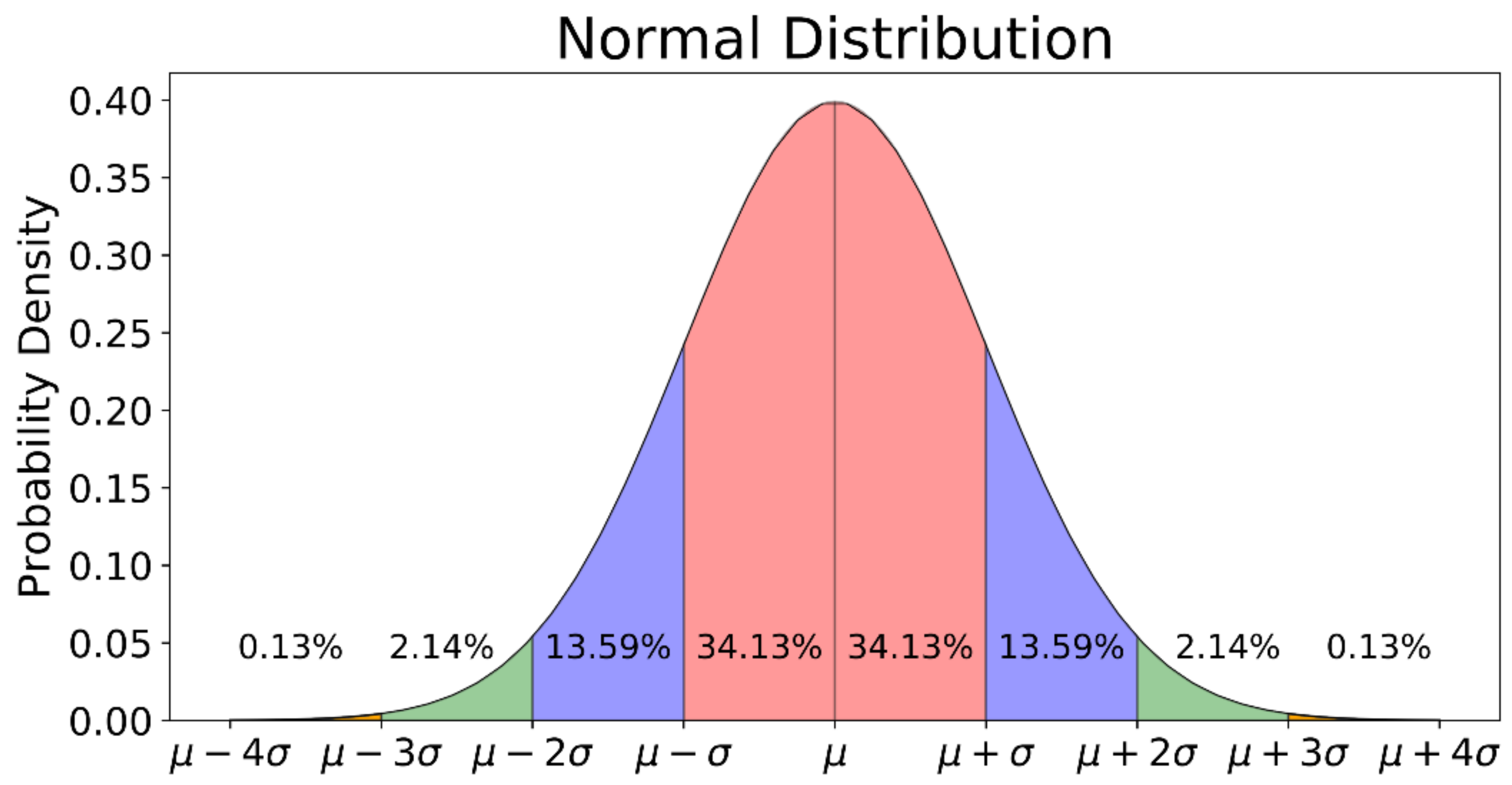
- Gauss described that the distribution of errors should satisfy three criteria:

    1. Small errors are more likely than large errors.

    2. For any real number ϵ the likelihood of errors of magnitude ϵ and $-\epsilon$ are equal.

    3. In the presence of several measurements of the same quantity, the most likely value of the quantity being measured is their average.

- We will now derive the Gaussian distribution using just these three criteria. (Buckle up!)



$\phi(x)$ "probability density function" "phi of x"

(1) $\phi(x)$ maximized when $x = 0$

(2) $\phi(x) = \phi(-x)$ ✓

Aside: new function $f(x) = \dfrac{\phi'(x)}{\phi(x)}$

Property of f: $\underbrace{f(-x)} = -f(x)$

$$f(-x) = \frac{\phi'(-x)}{\phi(-x)} = \frac{-\phi'(x)}{\phi(x)} = -f(x)$$

(3) Suppose p is some fixed, unknown quantity that we are trying to measure.

$\Rightarrow$ Let $M_1, M_2, \ldots, M_n$ be estimates/measurements of p.

$\Rightarrow$ Error: $M_1 - p, M_2 - p, \ldots, M_n - p$ (could be $p - M_i$ too)

$\Rightarrow$ Likelihood: $L(p) = \phi(M_1 - p) \cdot \phi(M_2 - p) \cdot \phi(M_3 - p) \cdots \phi(M_n - p)$

Told: the "most likely" p is average

$$\rightarrow \quad p = \frac{M_1 + M_2 + \cdots + M_n}{n}$$

$\rightarrow$ $L(p)$ maximized when $p = \overline{M}$

$$L(p) = \phi(M_1 - p) \cdot \phi(M_2 - p) \cdot \phi(M_3 - p) \cdots \phi(M_n - p)$$

$$\frac{d}{dp} L(p) = 0 \quad \text{when} \quad p = \bar{M}$$

$$\frac{d}{dp} L(p) = \underbrace{-\phi'(M_1 - p)\,\phi(M_2 - p)\,\phi(M_3 - p) \cdots \phi(M_n - p)}$$

$$\underbrace{-\phi(M_1 - p)\,\phi'(M_2 - p)\,\phi(M_3 - p) \cdots}$$

$$-\phi(M_1 - p)\,\phi(M_2 - p)\,\phi'(M_3 - p) \cdots$$

$$= -L(p) \cdot \frac{\phi'(M_1 - p)}{\phi(M_1 - p)} - L(p) \cdot \frac{\phi'(M_2 - p)}{\phi(M_2 - p)} - \cdots$$

$$a(x) = f(x)\, g(x)$$

$$a'(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

$$= a(x) \cdot \frac{f'(x)}{f(x)} + a(x) \cdot \frac{g'(x)}{g(x)}$$

$$\frac{d}{dp}L(p) = -L(p)\cdot\frac{\phi'(M_1-p)}{\phi(M_1-p)} - L(p)\cdot\frac{\phi'(M_2-p)}{\phi(M_2-p)} - \dots$$

$$= -L(p)\left[\sum_{i=1}^{n}\frac{\phi'(M_i-p)}{\phi(M_i-p)}\right]$$

$$= -L(p)\cdot\boxed{\sum_{i=1}^{n} f(M_i-p) = 0}$$

$p = \overline{M}$ satisfies

$$\sum_{i=1}^{n} f(M_i-\overline{M}) = 0$$

→ need an f that satisfies this for any measurements

$$\sum_{i=1}^{n} f(\mu_i - \mu) = 0 \quad \forall\, \mu_1, \mu_2, \ldots, \mu_n$$

$$f(kx) = k f(x)$$

$\Rightarrow$ key idea: f must be linear!
(linked reading)

$\Rightarrow \quad f(x) = cx \qquad$ c is some constant

$f(x) = cx$

$$\frac{\phi'(x)}{\phi(x)} = cx$$

integrate both sides

$$\int \frac{\phi'(x)}{\phi(x)}\,dx = \int cx\,dx$$

$$e^{\ln \phi(x)} = e^{\frac{c}{2}x^2 + D}$$

$$\int \frac{1}{u}\,du = \ln |u|$$

$$\phi(x) = e^{\frac{c}{2}x^2 + D}$$

$$= e^{\frac{c}{2}x^2} \cdot \underbrace{e^{D}}$$

$$= \boxed{Ae^{\frac{c}{2}x^2}}$$

① c must be -ve!

② $\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$

polar coordinates

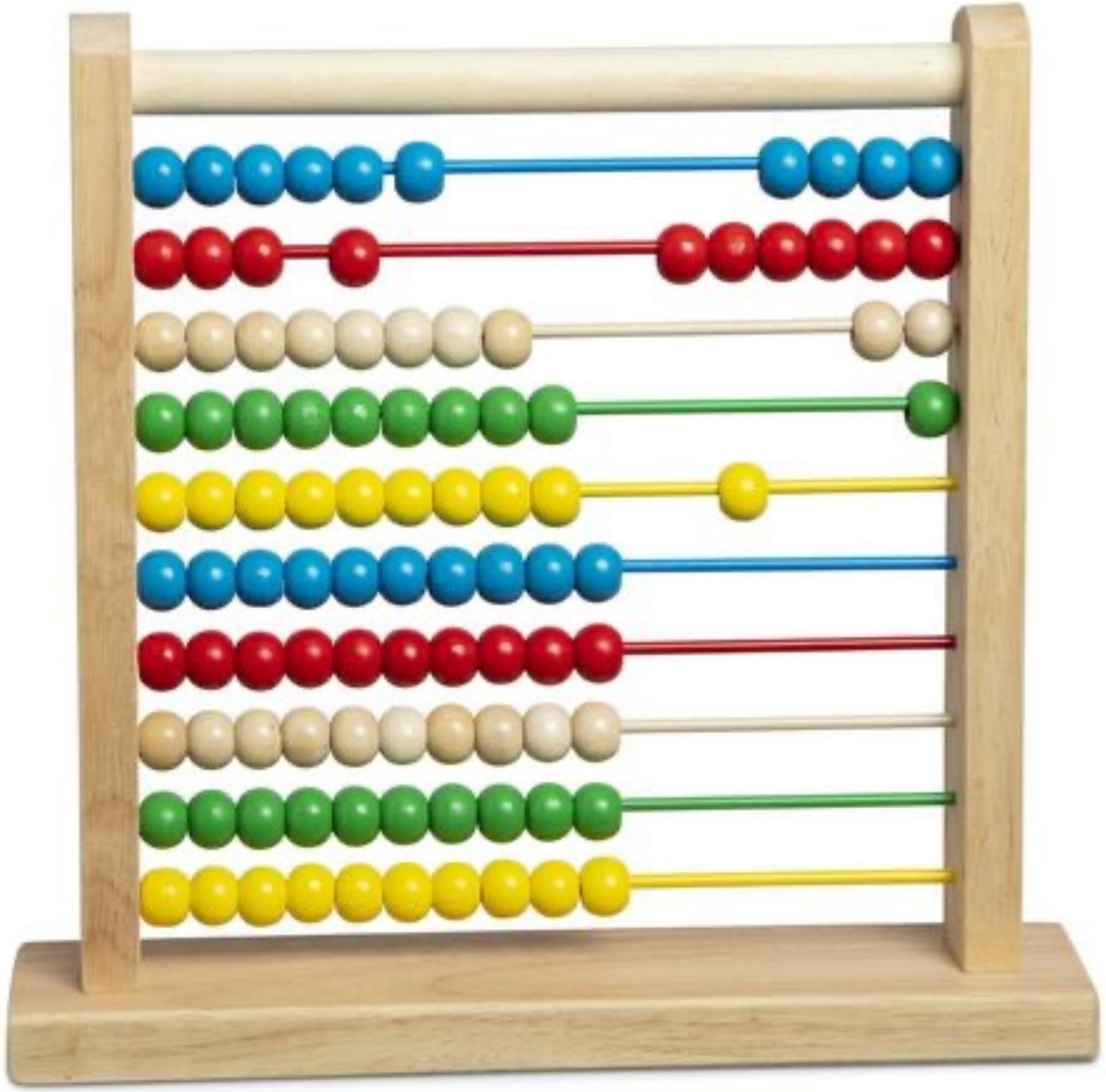
$$\left[\int_a^b f(x)\,dx\right]^2 = \int_a^b\int_a^b f(y)f(x)\,dx\,dy$$

Origins of computation

Abacus

3 12

- The **abacus** is an early form of a calculator, used in Ancient Mesopotamia and China as early as 1900 BC[1].
  - Abacuses are still in use today!
- It can be used to evaluate addition, subtraction, multiplication, and division without needing symbolic arithmetic.
  - The number system we use today is known as the Hindu-Arabic number system, and dates to ~700 AD.
- See [this site]() for a digital abacus.



1. https://criticallyconsciouscomputing.org/history

Computers

- In 1613[1], English poet Richard Braithwait published a book titled *The Yong Mans Gleanings*, which contains the earliest known reference to the word "computer".
- In it, a "computer" was defined as a **person** who performed arithmetic and algebraic operations.
- "Computer"s referred to people as recent as the mid-1900s.
    - Then, most computers were women, because they could be paid less than men.

1. https://criticallyconsciouscomputing.org/history

Leibniz and binary

- Gottfried Wilhelm Leibniz (1~~4~~664-1716) – one of the “creators of calculus” that we studied earlier in the course – is also credited for developing the **binary number system**.
    - There are two digits in binary – 1 and 0. “Bit” stands for “binary digit.”
- At the time, he had no practical use for it, but in modern computing all code is represented at a low-level in binary.
    - Transistors – the building block of computer processors – are like switches that can either be ”on” (1) or ”off” (0).
- One of this week’s readings contains a translation of his original work that discussed binary numbers, titled *“EXPLANATION OF BINARY ARITHMETIC, WHICH USES ONLY THE CHARACTERS 0 AND 1, WITH SOME REMARKS ON ITS USEFULNESS, AND ON THE LIGHT IT THROWS ON THE ANCIENT CHINESE FIGURES OF FU XI”*.

Number systems

decimal

dec = 10

- Our standard number system, base 10, has 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

  3429_{10}

  - We can decompose the number ~~3249~~$_{10}$ as follows:

$$3429_{10} = 3 \cdot 10^3 + 4 \cdot 10^2 + 2 \cdot 10^1 + 9 \cdot 10^0$$

- Similarly, base 2 (i.e. binary) only has 2 digits: 0 and 1.
  - To convert from binary to base 10, we can follow a similar procedure.

$$1001_{\cancel{10}\,2} = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 1 = 9$$

Converting between binary and base 10

decimal

Example: Convert 324_{10} to binary.

$324 = 256 + 64 + 4$

$= 2^8 + 2^6 + 2^2$

$$\begin{array}{r} 324 \\ -\ 256 \\ \hline 68 \\ -\ 64 \\ \hline 4 \end{array}$$

$\boxed{101000100_2}$

Example: Convert 110001011_2 to base 10.

8 7 6 5 4 3 2 1 0

$= 2^8 + 2^7 + 2^3 + 2^1 + 2^0$

$= \boxed{395}$

Binary arithmetic

- Arithmetic in binary largely works the same way that arithmetic in base 10 works.
- **Example:** multiply 10110_2 and 11_2.

yellkey.com/adult

word: adult

addition

$$\begin{array}{r} 1011_2 \\ 1010_2 \\ \hline 10101_2 \end{array}$$

$1011_2 \to 2^0 + 2^1 + 2^3 = 11$

$1010_2 \to 10$

$10101_2 \to 2^0 + 2^2 + 2^4 = \boxed{21}$

$$\begin{array}{r} 10110 \\ \times \quad 11 \\ \hline 10110 \\ 10110\mathrm{x} \\ \hline 1000010 \end{array}$$

Symbols

- We've looked at how base 10 numbers can be stored in binary.
- But base 10 numbers are not the only thing our computers need to store and work with.
  - What about negative numbers? Decimals?
  - Strings?
  - Colors?
  - All of these can be stored in binary as well.

Boolean algebra

- George Boole (1815-1864) was an English mathematician.
- In 1854, he published *An investigation into the Laws of Thought, on Which are founded the Mathematical Theories of **Logic** and Probabilities*, in which he laid the foundations of **Boolean algebra**.
  - In Boolean algebra, there are two values – 1 (True) and 0 (False), and three operators – AND, OR, and NOT.
- Fun fact: at the age of 19, Boole created his own elementary school!

1st. Disjunctive Syllogism.

Either X is true, or Y is true (exclusive),	$x + y - 2xy = 1$
But X is true,	$x = 1$
Therefore Y is not true,	$\therefore y = 0$
Either X is true, or Y is true (not exclusive),	$x + y - xy = 1$
But X is not true,	$x = 0$
Therefore Y is true,	$\therefore y = 1$

2nd. Constructive Conditional Syllogism.

If X is true, Y is true,	$x(1-y) = 0$
But X is true,	$x = 1$
Therefore Y is true,	$\therefore 1 - y = 0$ or $y = 1$.

3rd. Destructive Conditional Syllogism.

If X is true, Y is true,	$x(1-y) = 0$
But Y is not true,	$y = 0$
Therefore X is not true,	$\therefore x = 0$

4th. Simple Constructive Dilemma, the minor premiss exclusive.

If X is true, Y is true,	$x(1-y) = 0$, (41),
If Z is true, Y is true,	$z(1-y) = 0$, (42),
But Either X is true, or Z is true,	$x + z - 2xz = 1$, (43).

From the equations (41), (42), (43), we have to eliminate x and z. In whatever way we effect this, the result is

$$y = 1;$$

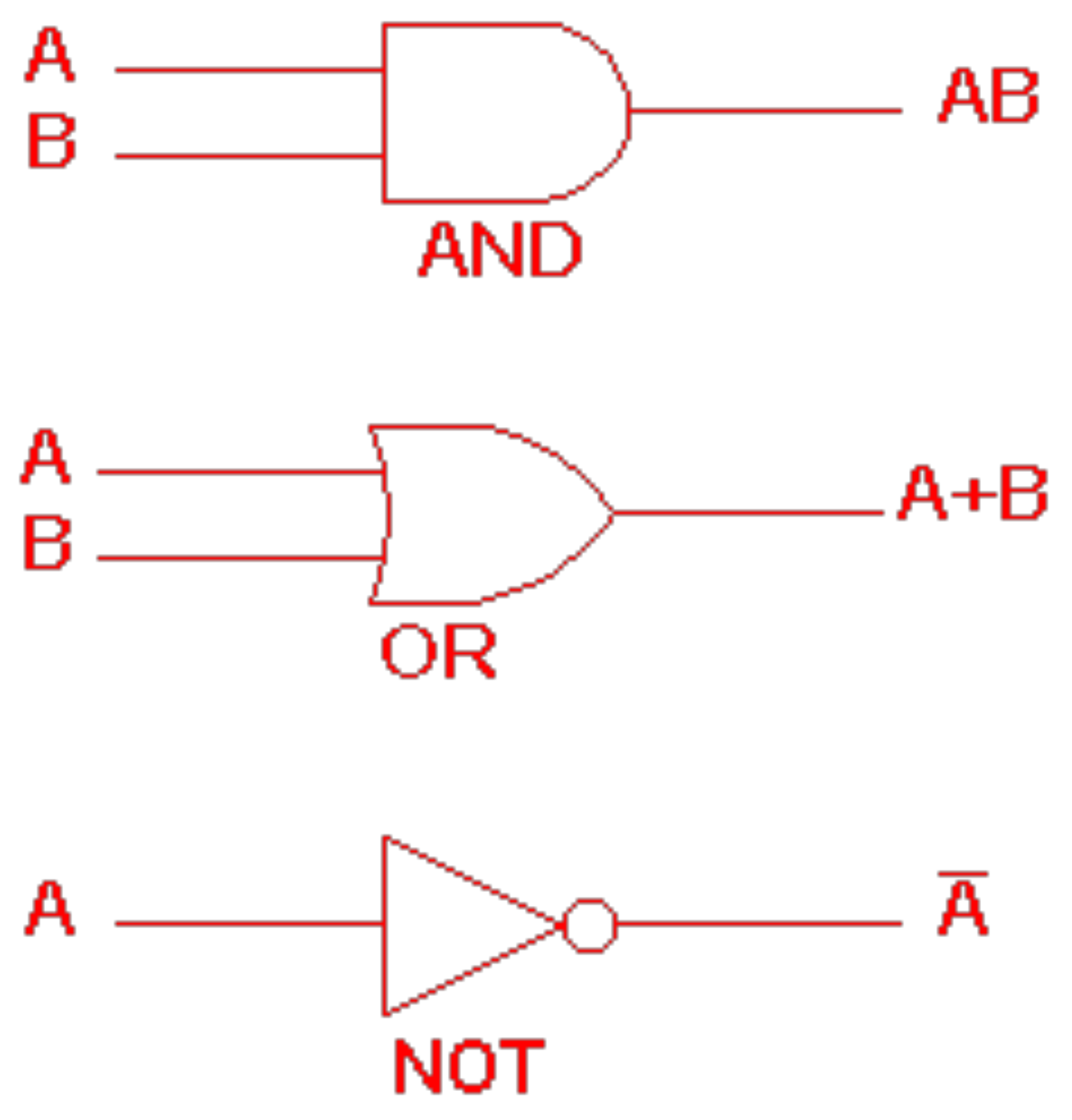
whence it appears that the Proposition Y is true.

Boolean algebra

- In Boolean algebra, there are two values – 1 (True) and 0 (False), and three operators – AND, OR, and NOT.

- All other operations can be constructed using a combination of these three operators.

- Circuits use Boolean algebra to control the flow of current.



That’s all!